EQUIVARIANT MORSE THEORY FOR FLOWS
AND AN APPLICATION TO THE N-BODY PROBLEM

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ABSTRACT. In this paper, using Conley's index and equivariant cohomology, some Morse type inequalities are deduced for a flow equivariant with respect to the action of a compact topological group.

In the case of a gradient flow induced by a nondegenerate smooth function these inequalities coincide with those described by R. Bott.

The theory is applied to the study of the central configurations of N-bodies.

Introduction. When a flow on a topological space is subject to some symmetry there are at least three ways of obtaining Morse inequalities [16]. With simple examples it is possible to prove that, in the case of a nonfree action of a symmetry group, these three approaches give different information [16].

In this paper we describe the equivariant Morse theory for flows in the presence of a symmetry due to the action of a compact topological group.

The theory presented here generalizes the one described in [1 and 4] for a gradient flow induced by a nondegenerate smooth function. In particular, our extension allows us to treat the case of functions having degenerate critical orbits.

We point out that a different approach to the study of equivariant gradient flows, in the presence of particular symmetries, is described in [2, 9, 10] using the cohomological or geometrical index as a replacement for the Ljusternik-Schnirelmann category.

In §3 we apply the equivariant Morse theory to the N-body problem, showing that the Morse relations obtained in [15] hold also when the potential energy has some degenerate critical orbits.

Moreover we prove that the Morse inequalities for the spatial problem imply the existence of *many more central configurations* than one could deduce from the Morse inequalities for the planar problem [14]. This is relevant for the question of the existence of spatial central configurations.

We end §3 by proving that the collinear problem is a repeller in the gradient flow induced by the potential energy.

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1. Preliminaries. Let Γ be a Hausdorff topological space and the map $(\gamma, t) \to \gamma \cdot t$ from $\Gamma \times \mathbf{R}$ to Γ a flow on Γ [7, 8, 16, 17].

We recall [8] that a locally compact Hausdorff subspace X of Γ is a local flow if for every $\gamma \in X$ there is a neighborhood $U \subset \Gamma$ of γ and an $\varepsilon > 0$ such that $(X \cap U) \cdot [0, \varepsilon) \subset X$.

We suppose that there is a (left) action of a compact topological group G on Γ [6, 16].

A subset $Y \subseteq \Gamma$ is called G-invariant if

$$g \in G \quad \text{and} \quad \gamma \in Y \Rightarrow g\gamma \in Y$$

where $g\gamma$ denotes the element of Y obtained from γ using the action of $g \in G$.

We say that the flow on Γ is *equivariant* if

$$(1.1) \qquad (g\gamma) \cdot t = g(\gamma \cdot t), \qquad \gamma \in \Gamma, \, g \in G, \, t \in \mathbf{R}.$$

This allows us to define a flow on the quotient Γ/G in the following way:

$$(1.2) \qquad [\gamma] \cdot t = [\gamma \cdot t], \qquad [\gamma] \in \Gamma/G, \, t \in \mathbf{R},$$

where $[\gamma]$ is the orbit of a point $\gamma \in \Gamma$ under the action of G.

The flow (1.2) is well defined because if γ' belongs to the orbit of γ, then $\gamma' = g\gamma$, for some $g \in G$, and consequently

$$[\gamma'] \cdot t = [\gamma' \cdot t] = [(g\gamma) \cdot t] = [g(\gamma \cdot t)] = [\gamma \cdot t] = [\gamma] \cdot t.$$

It is easy to prove the following proposition.

PROPOSITION 1.1. *Let X be a G-invariant local flow in the G-equivariant flow on Γ and I a G-invariant isolated invariant set in X. Then*
(i) X/G is a local flow in Γ/G,
(ii) I/G is an isolated invariant set in X/G,
(iii) if (N, N^-) is an index pair for I with N and N^- G-invariant then $(N/G, N^-/G)$ is an index pair for I/G.

PROOF. (i) X/G is locally compact and is an Hausdorff subspace of Γ/G because G is a compact topological group [6].

Take $[\gamma] \in X/G$, $\gamma \in X$. Since X is a local flow in Γ, there exists a neighborhood $U_\gamma \subset \Gamma$ and $\varepsilon > 0$ such that $(X \cap U_\gamma) \cdot [0, \varepsilon) \subset X$.

Since G is compact any neighborhood of a G-invariant set contains a G-invariant neighborhood [6, Ex. 9, p. 69], so we can suppose that U_γ is G-invariant. Then U_γ/G is a neighborhood of $[\gamma]$ in Γ/G and $(X/G \cap U_\gamma/G) \cdot [0, \varepsilon) \subset X/G$.

(ii) To say that I is an isolated invariant set in X means [7, 8] that there exists a compact $N \subset X$ such that $I \subset \text{int } N$ (relative to X) and $I = I(N)$ where $I(N)$ is the maximal invariant set contained in N, that is $I(N) = \{\gamma \in N \,|\, \gamma \cdot \mathbf{R} \subset N\}$.

As above we can suppose that N is G-invariant. Then $I/G \subset \text{int}(N/G)$, N/G is compact and $I/G = \{[\gamma] \in N/G \,|\, [\gamma] \cdot \mathbf{R} \subset N/G\}$.

(iii) We recall that a compact pair (N, N^-) is an index pair for the isolated invariant set I [8] if

(a) $\mathrm{cl}(N \setminus N^-)$ is an isolating neighborhood of I,

(b) $\gamma \in N^-$ and $\gamma \cdot [0, t] \subset N$ imply $\gamma \cdot [0, t] \subset N^-$,

(c) if $\gamma \in N$ and $\gamma \cdot \mathbf{R}^+ \not\subset N$ then there exists $\bar{t} \geq 0$ such that $\gamma \cdot [0, \bar{t}] \subset N$ and $\gamma \cdot \bar{t} \in N^-$.

Since N and N^- are G-invariant and compact, $N \setminus N^-$ is G-invariant and relatively compact. Thus, by (ii) $\mathrm{cl}(N/G \setminus N^-/G) = \mathrm{cl}((N \setminus N^-)/G) = (\mathrm{cl}(N \setminus N^-))/G$ is an isolating neighborhood of I/G.

Moreover, if $\alpha \in N^-/G$, then $\alpha = [\gamma]$ for some $\gamma \in N^-$.

From $\alpha \cdot [0, \bar{t}] \subset N/G$ and (b) we have $\forall t \in [0, \bar{t}]$

$$[\gamma] \cdot t = [\gamma \cdot t] \in N/G \Rightarrow \gamma \cdot t \in N$$
$$\Rightarrow \gamma \cdot t \in N^- \Rightarrow [\gamma \cdot t] = [\gamma] \cdot t \in N^-/G.$$

Finally, if $\alpha \in N/G$ and $\alpha \cdot \mathbf{R}^+ \not\subset N/G$, then $\alpha = [\gamma]$, for some $\gamma \in N$ and there exists $t \in R^+$ such that $[\gamma] \cdot t = [\gamma \cdot t] \notin N/G$. This implies that $\gamma \cdot \mathbf{R}^+ \not\subset N$. Thus, from (c) there exists $\bar{t} \geq 0$ such that $\forall t \in [0, \bar{t}]$, $\gamma \cdot t \in N$ and $\gamma \cdot \bar{t} \in N^-$. Consequently $\alpha \cdot t = [\gamma \cdot t] \in N/G$, $\forall t \in [0, \bar{t}]$, and $\alpha \cdot \bar{t} = [\gamma \cdot \bar{t}] \in N^-/G$. $\square$

REMARK 1.1. Given the G-invariant isolated invariant set I it is always possible to construct an index pair (N, N^-) with N and N^- G-invariant. In fact, since I/G is an isolated invariant set in X/G, there exists an index pair (M, M^-) of I/G in X/G. Then, denoted by π the canonical projection of Γ on Γ/G, $(\pi^{-1}(M), \pi^{-1}(M^-))$ is an index pair for I in X.

From now on we will denote by S a G-invariant isolated invariant set in the G-invariant local flow X.

PROPOSITION 1.2. *If $(M_1, \ldots, M_n)$ is an admissible ordering of a Morse decomposition of S, given by G-invariant Morse sets, then $(M_1/G, \ldots, M_n/G)$ is an admissible ordering of a Morse decomposition of S/G.*

PROOF. A Morse decomposition of S [7, 8] is a finite collection $\{M_\pi\}_{\pi \in P}$ of disjoint, compact, invariant subsets $M_\pi \subset S$ which can be ordered $(M_1, \ldots, M_n)$ in such a way that for every $\gamma \in S \setminus \bigcup_{1 \leq j \leq n} M_j$ there are indices $i < j$ such that $\omega(\gamma) \subset M_i$ and $\omega^*(\gamma) \subset M_j$.

The ω-limit sets, $\omega(\gamma)$ and $\omega^*(\gamma)$ are defined to be, respectively, the maximal invariant set in the closure of $\gamma \cdot [0, +\infty)$ and $\gamma \cdot (-\infty, 0]$.

Since each M_j is G-invariant and G is compact it is easy to see that $\{M_\pi/G\}_{\pi \in P}$ is a finite collection of disjoint, compact, invariant subsets of S/G.

Now let us take for $\{M_\pi/G\}_{\pi \in P}$ the same ordering as for $\{M_\pi\}_{\pi \in P}$ and prove that this ordering is admissible.

To do this it is enough to show that if γ and γ' are in the same equivalence class, that is, $\gamma' = g\gamma$, for some $g \in G$, then

(1.3) $$\omega(\gamma') = \omega(g\gamma) = g\omega(\gamma) \quad \text{and} \quad \omega^*(\gamma') = g\omega^*(\gamma).[1]$$

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[1] Given $A \subset \Gamma$, we denote by gA, $g \in G$, the set $gA = \{gx, x \in A\}$.

In fact, if $\alpha \in S/G \setminus \bigcup_{1 \leqslant i \leqslant n} M_i/G$,

$$\omega(\alpha) = \left(\bigcup_{g \in G} \omega(g\gamma) \right) \Big/ G,$$

with γ such that $[\gamma] = \alpha$, and so (1.3) together with the G-invariance of each M_π implies the assertion.

Let us prove (1.3). By definition $\omega(g\gamma)$ is the maximal invariant set contained in $\mathrm{cl}(\{ g\gamma \cdot [0, \infty)\})$.

Then we have

$$\omega(g\gamma) = I\big(\mathrm{cl}(\{ g\gamma \cdot [0, \infty)\})\big) = I\big(\mathrm{cl}(\{ g(\gamma \cdot [0, \infty))\})\big)$$
$$= I\big(g\,\mathrm{cl}(\{ \gamma \cdot [0, \infty)\})\big) = gI\big(\mathrm{cl}(\{ \gamma \cdot [0, \infty)\})\big) = g\omega(\gamma).$$

In the same way it is possible to prove that $\omega^*(g\gamma) = g\omega^*(\gamma)$. □

REMARK 1.2. If S is G-invariant and $(M_1, \ldots, M_n)$ is a Morse decomposition of S, it is possible to obtain a new Morse decomposition of S, built up by G-invariant Morse sets, putting together the sets M_i which are in the same orbit.

Actually, if the group is connected, each isolated invariant set must be G-invariant.

2. Equivariant Morse theory. As in §1 we suppose that an equivariant flow is defined on the Hausdorff topological space Γ.

If G is a compact topological group there exists a universal G-bundle characterized by having its total space E contractible:

$$\begin{array}{c} G \\ \downarrow \\ E \\ \downarrow \\ E/G = BG \end{array}$$

(2.1)

The space BG is the so-called classifying space of G. The action of G on E is free[2] and E is unique, up to homotopy.

Since the action of G on E is free, the diagonal action of G on the product $\Gamma \times E$ defined by

$$g(\gamma, e) = (g\gamma, ge), \qquad g \in G, \gamma \in \Gamma, e \in E,$$

is also free.

The flow can be extended to the product $\Gamma \times E$ in the trivial way:

$$(\gamma, e) \cdot t = (\gamma \cdot t, e), \qquad t \in \mathbf{R}.$$

Of course this flow is G-equivariant on the product $\Gamma \times E$. Then it can be projected to a flow on the quotient space $(\Gamma \times E)/G = \Gamma_G$.

It is obvious that if I is a G-invariant, invariant set for the flow on Γ, then $(I \times E)/G = I_G$ is an invariant set for the quotient flow in Γ_G.

The aim of this section is to obtain a kind of "Morse inequality" for this quotient flow.

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[2]An action of G on a space T is said to be free if $g \in G$ and $g \neq 1 \Rightarrow gx \neq x$ $\forall x \in T$.

To do this, following the procedure of [8], we need a filtration of compact pairs. But, usually, in the universal bundle (2.1), E and BG are realized as infinite dimensional manifolds, hence we cannot hope to construct compact pairs for any isolated invariant set.

Actually the definition itself of isolated invariant set requires the presence of a compact isolating neighborhood, so that we cannot even say that a set is an isolated invariant set in Γ_G.

Using a limit procedure we can still derive some "Morse inequalities".

When G is a compact topological group, E and BG can be obtained as the limit of finite dimensional compact spaces

$$E = \lim_{k \to \infty} E_k, \quad BG = \lim_{k \to \infty} B_k G,$$

related to the bundles:

$$G$$
$$\downarrow$$
$$E_k$$
$$\downarrow$$
$$E_k / G = B_k G$$

The action of G on E_k is free and hence the diagonal action on the product $\Gamma \times E_k$ is also free.

As before we extend the flow to the product $\Gamma \times E_k$ in the trivial way and we project it to the quotient $(\Gamma \times E_k)/G$.

Then we have

PROPOSITION 2.1. *If I is a G-invariant isolated invariant set for the flow on Γ, then*
(a) *$(I \times E_k)/G$ is an isolated invariant set for the flow on $(\Gamma \times E_k)/G$.*
(b) *If (N, N^-) is an index pair for I and N and N^- are G-invariant, then $((N \times E_k)/G, (N^- \times E_k)/G) = (N_k, N_k^-)$ is an index pair for $(I \times E_k)/G$.*
(c) *If $(M_1, \ldots, M_n)$ is an admissible ordering of a Morse decomposition of I, given by G-invariant Morse sets, then $((M_1 \times E_k)/G, \ldots, (M_n \times E_k)/G)$ is an admissible ordering of a Morse decomposition of $(I \times E_k)/G$.*

PROOF. (a) and (b) follow from (ii) and (iii) of Proposition 1.1 because it is easy to see that $I \times E_k$ is an isolated invariant set in $\Gamma \times E_k$ and $(N \times E_k, N^- \times E_k)$ is a G-invariant index pair for $I \times E_k$.

Part (c) follows from Proposition 1.2 because $(M_1 \times E_k, \ldots, M_n \times E_k)$ is an admissible ordering of a Morse decomposition of $(I \times E_k)$. $\square$

If E_k is a Hausdorff space the previous proposition can be formulated for the local flow $(X \times E_k)/G$.

Let us denote by S a G-invariant isolated invariant set in Γ and by $(M_1, \ldots, M_n)$ a Morse decomposition of S built up by G-invariant sets.

By virtue of Proposition 2.1, for each $k \in N$ we can write the following Morse inequalities [8]:

(2.2)
$$\sum_{j=1}^{n} P_t\big(h_k(M_j)\big) = P_t\big(h_k(S)\big) + (1 + t)Q_t^k$$

where $h_k(S)$ and $h_k(M_j)$ are, respectively, the (homotopy) index of $(S \times E_k)/G$ and $(M_j \times E_k)/G$, $P_t(\cdot)$ is the Poincaré series which represents the Čech-cohomology of any element in the equivalence class $h_k(S)$ or $h_k(M_j)$ and Q_t^k is a series with nonnegative integer coefficients.

We recall that $P_t(h_k(S))$ (resp. $P_t(h_k(M_j))$) can be computed from any index pair $((N \times E_k)/G, (N^- \times E_k)/G)$ of $(S \times E_k)/G$ (resp. $(M_j \times E_k)/G$) if (N, N^-) is a G-invariant index pair for S (resp. M_j).

Now, if we take the limit in (2.2), $k \to \infty$, using the stabilization property of the cohomology for the classifying spaces of compact topological groups [11, Chapter III][3] we obtain

$$(2.3) \qquad \sum_{j=1}^{n} P_t^G(h(M_j)) = P_t^G(h(S)) + (1 + t)Q_t^G.$$

In (2.3) $P_t^G(h(S))$ (resp. $P_t^G(h(M_j))$) represents the cohomology of the pair $((N \times E)/G, (N^- \times E)/G)$, if (N, N^-) is a G-invariant index pair for S (resp. M_j), that is the equivariant cohomology of the pair (N, N^-).[4]

If I is a G-invariant isolated invariant set and (N, N^-) is a G-invariant index pair for I, the homotopy type of the pair $((N \times E)/G, (N^- \times E)/G)$, will be denoted by $h_G(I)$ and called the equivariant-(homotopy) index of I.

Thus (2.3) could be written

$$(2.4) \qquad \sum_{j=1}^{n} P_t(h_G(M_j)) = P_t(h_G(S)) + (1 + t)Q_t^G.$$

PROPOSITION 2.2. *If I is a G-invariant isolated invariant set and the action of G on I is free then $P_t(h_G(I)) = P_t(h(I/G))$ where $h(I/G)$ is the (homotopy) index of I/G in the quotient flow on Γ/G.*

PROOF. Since the action on I is free and the set of the points in Γ on which the action is free is open in Γ it is possible to find an open neighborhood V of I in Γ on which the action is also free.

Then we can take an index pair (N, N^-) with $N \subset V$, so that the action of G is free on N. But, whenever the action on a space is free the equivariant cohomology coincides with the ordinary cohomology of the quotient space. So we have

$$P_t(h_G(I)) = P_t((N \times E)/G, (N^- \times E)/G)$$
$$= P_t(N/G, N^-/G) = P_t(h(I/G)). \qquad \square$$

We close this section by showing that if G is a compact Lie group and we have a gradient flow on a G-invariant compact manifold M, induced by a nondegenerate G-invariant smooth function f (2.3) generalizes the (classical) equivariant Morse inequalities described in [1 and 4].

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[3] If $E = \lim_{k \to \infty} E_k$ and $BG = \lim_{k \to \infty} E_k/G$ then for each $i \in N$ there exists $m(i) \in N$, such that $k \geqslant m(i) \Rightarrow H^i(E) \cong H^i(E_k)$ and $H^i(BG) \cong H^i(E_k/G)$.

[4] If G acts on a space X and E is the total space in the bundle (2.1) the equivariant cohomology of X, $H_G(X)$, is, by definition, the cohomology of $(X \times E)/G = X_G$.

In fact, considering again the bundle (2.1) the function f can be extended to a G-invariant function on $M \times E$ and hence projected to a function f_E on $(M \times E)/G = M_G$.

The important thing is that in doing this we do not lose the nondegeneracy of f, that is f_E is still nondegenerate on M_G. Moreover, if T is a nondegenerate critical manifold[5] of f on M, $(T \times E)/G$ is a nondegenerate critical manifold of f_E and the Morse indices of T relative to f and $(T \times E)/G$ relative to f_E are equal [1].

Since a nondegenerate G-invariant function has only a finite number of critical orbits, which are nondegenerate critical manifolds, we can take as a Morse decomposition of f the one given by these critical orbits, ordering them according to the values of f [8, 16].

From (2.3) we get

$$(2.5) \qquad \sum_T P_t^G(h(T)) = P_t^G(M) + (1 + t)Q_t^G$$

where the summation is extended to all the critical orbits of f.

Because of the lack of compactness (2.5) is obtained using the same approximation method employed in deriving (2.3). That is $P_t^G(h(T)) = \lim_{k \to \infty} P_t(h_k(T))$.

But for nondegenerate critical manifolds the (homotopy) index can be computed more directly [1, 16]:

$$(2.6) \qquad P_t(h_k(T)) = t^{\lambda_T} P_t((T \times E_k)/G, \mathcal{O}^- \otimes K)$$

where λ_T is the number of negative eigenvalues of the hessian of f in the direction normal to $(T \times E_k)/G$ (which is the same as for T), K is a ring and $P_t(\cdot, \mathcal{O}^- \otimes K)$ denotes the cohomology with local coefficients. More precisely $\mathcal{O}^-$ is the orientation bundle of $\nu^-(T)$, which is the part of the normal bundle $\nu(T)$ spanned by the negative eigenvalues of the hessian of f on $\nu(T)$. In particular, if $\nu^-(T)$ is orientable $P_t(T, \mathcal{O}^- \otimes K)$ is the cohomology with ordinary coefficients.

Then passing to the limit in (2.6), for each critical orbit T, (2.5) becomes

$$(2.7) \qquad \sum_T t^{\lambda_T} P_t^G(T, \mathcal{O}^- \otimes K) = P_t^G(M) + (1 + t)Q_t^G$$

where $P_t^G(M)$ (resp. $P_t^G(T)$) represents the equivariant cohomology of M (resp. T).

Moreover each T, being a single orbit, is homeomorphic to G/H, where H is the isotropy group[6] of each point of T. Then we have

$$(T \times E)/G \cong (G/H \times E)/G \cong E/H \cong BH.$$

The last relation holds because if E is the total space for a universal G-bundle, E is also the total space for a universal H-bundle; hence E/H is homotopically equivalent to BH, the classifying space of H.

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[5] T is a nondegenerate critical manifold of f if [5]
   (i) each point $p \in T$ is a critical point of f.
   (ii) the hessian of f is nondegenerate in the normal direction to T.
[6] The isotropy group of a point $x \in T$ is the subgroup of G defined by $G_x = \{g \in G \mid gx = x\}$.

Then (2.7) becomes

$$(2.8) \qquad \sum_T t^{\lambda_T} P_t(BH, \mathcal{O}^- \times K) = P_t^G(M) + (1 + t)Q_t^G.$$

Moreover, if H is connected, we do not need to use local coefficients because, in this case, BH is simply connected and every bundle over a simply connected base is orientable.

3. An application to the N-body problem. Let us denote by $q = (q_1, \ldots, q_N) \in \mathbf{R}^{3N}$ the positions of N bodies with masses $m_1, \ldots, m_N$ respectively.

The potential energy is the function

$$V(q) = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}.$$

Keeping the same notations as in [15] we denote by $\mathcal{E}$ the manifold

$$\mathcal{E} = \left\{ q \in \mathbf{R}^{3N}: \sum_{i=1}^{N} m_i q_i = 0 \text{ and } \sum_{i=1}^{N} m_i |q_i|^2 = 1 \right\}$$

and by Δ the set $\{ q \in \mathbf{R}^{3N}: \text{ there exists } i \neq j \text{ such that } q_i = q_j \}$.

We are interested in studying the critical points of V restricted to the manifold $M = \mathcal{E} \setminus \Delta$.

As shown in [15], this is equivalent to studying the central configurations of N bodies.

We also consider the cases when the bodies lay on a fixed line or on a fixed plane. Let us examine both cases.

When the bodies are constrained to move on a fixed line the problem is called collinear. In this case the manifold T corresponding to $\mathcal{E} \setminus \Delta$ is $N - 2$ dimensional and is invariant, together with the potential V, under the diagonal action of the group Z_2, given by the reflection with respect to the origin.

If the bodies are constrained to move on a fixed plane the problem is called coplanar and the manifold Y corresponding to $\mathcal{E} \setminus \Delta$ is $2N - 3$ dimensional. The problem is invariant under the action of the group $O(2)$, given by the rotations in that plane plus the reflections with respect to the lines through the origin.

In this case any collinear configuration has a nontrivial isotropy group isomorphic to Z_2, while the action is free on the planar configurations. If we restrict our attention only to the action of $SO(2) = S^1$ on Y, then the action is free also on the collinear configurations.

The coplanar problem has been studied by Palmore by applying Morse theory to the quotient space Y/S^1 [14].

If there is no restriction on the motion of the N bodies the problem is called spatial.

The manifold M is $3N - 4$ dimensional and is invariant for the action of $O(3)$, as well as the potential $V(q)$.

The nontrivial isotropy groups are $O(2)$ and Z_2 for the collinear and planar configurations, respectively.

To consider only the action of $SO(3) \subset O(3)$ does not make the action free but reduces the isotropy groups to S^1 and the identity, respectively.

The spatial problem has been studied in [15] by applying the equivariant Morse theory to the $SO(3)$-equivariant flow $\dot{q} = \nabla V \mid_M$.

When the masses $(m_1, \dots, m_N)$ are such that the corresponding potential energy has only nondegenerate critical orbits the equivariant Morse relations obtained in [15] are

(3.1)
$$
\left\{
\begin{aligned}
\sum_{\lambda=0}^{2N-5} (\alpha_\lambda + \beta_\lambda) t^\lambda + \left(\frac{N!}{2}\right) \frac{t^{2N-4}}{1-t^2} &= \frac{(1+t^2)\cdots(1+(N-1)t^2)}{1-t^4} + Q_t(V) \\
&= \frac{1}{1-t^2} \sum_{i=0}^{N-2} \gamma_{2i} t^{2i} + (1+t)Q_t(V)
\end{aligned}
\right.
$$

where

(i) α_λ is the number of spatial critical orbits, whose points have index λ,[7]

(ii) β_λ is the number of planar critical orbits whose points have index λ,

(iii) $(N!/2)t^{2N-4}/(1-t^2)$ is the part related to the collinear configurations (according to (2.8)),

(iv) $\sum_{i=0}^{N-2} \gamma_{2i} = N!/2$.

From (3.1) the following estimate holds [15]:

(3.2) $\alpha_{2j} + \beta_{2j} \geqslant \gamma_0 + \gamma_2 + \cdots + \gamma_{2j}, \qquad 0 \leqslant 2j \leqslant 2N - 5.$

As a consequence of Proposition 2.2 we have

PROPOSITION 3.1. *Equivariant Morse relations of* (3.1) *type hold also when the potential energy has a finite number of degenerate critical orbits.*

PROOF. Since the action of $SO(3)$ is free on the spatial and planar configurations the equivariant index of each critical orbit coincides with the index in the quotient space $M/SO(3)$.

Then this index is expressed by a polynomial of the type $P(t) = \sum_{i=0}^{2N-5} a_i t^i$; so that summing up all the polynomials we obtain the expression (3.1) where, of course, λ does not represent the number of positive eigenvalues of the hessian of V any more. $\square$

The previous proposition implies that if each $P(t)$ is a monomial $\alpha_\lambda t^\lambda$ (or $\beta_\lambda t^\lambda$), from (3.2) we deduce an estimate of the number of spatial and planar central configurations, *even in the presence of degenerate critical orbits.*

Now let us compare (3.1) with the Morse inequalities for the planar problem [14].

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[7] Here the word *index* means the number of positive eigenvalues of the hessian of V at each point of a critical orbit.

Applying Morse theory in the quotient manifold Y/S^1 one gets
(3.3)

$$
\begin{cases}
\displaystyle\sum_{i=0}^{N-2} b_i t^i + \left(\frac{N!}{2}\right) t^{N-2} = (1 + 2t)(1 + 3t) \cdots (1 + (N-1)t) + (1 + t)Q_t(V) \\
\\
\qquad\qquad\qquad = \displaystyle\sum_{i=0}^{N-2} \delta_i t^i + (1 + t)Q_t(V),
\end{cases}
$$

$$
\sum_{i=0}^{N-2} = \frac{N!}{2}.
$$

From (3.3), equating the coefficients of the same powers and denoting by $2p$ the number of S^1-critical orbits of planar configurations (except the collinear ones) we get

$$
2p \geqslant 1 + \delta_1 + \cdots + \delta_{N-3} + (N!/2 - \delta_{N-2}).
$$

Since $N!/2 = 1 + \delta_1 + \cdots + \delta_{N-2}$, we have

(3.4)
$$
2p \geqslant 2(1 + \delta_1 + \cdots + \delta_{N-3}).
$$

But, as we observed before, the coplanar problem is invariant under the $O(2)$-symmetry and $O(2)$ is contained in $SO(3)$. Then the number of $SO(3)$-critical orbits coming from planar configurations is

(3.5)
$$
p \geqslant 1 + \delta_1 + \cdots + \delta_{N-3} = m.
$$

From (3.2), denoting by s the number of $SO(3)$-critical orbits of spatial configurations (i.e. except the collinear and coplanar ones) we get

(3.6)
$$
s + p \geqslant 1 + (1 + \gamma_2) + \cdots + (1 + \gamma_2 + \cdots + \gamma_{2N-6}) = \mu.
$$

Since $\delta_i = \gamma_{2i}$, $0 \leqslant i \leqslant N - 3$, we have

(3.7)
$$
\mu - m \geqslant 1 + (1 + \delta_1) + \cdots + (1 + \delta_1 + \cdots + \delta_{N-4})
$$

so that $\mu - m \to \infty$ for $N \to \infty$.

The estimate (3.7) is very important because it shows that from the equivariant Morse inequalities one deduces the presence of *many more central configurations than one could guess from the Morse inequalities for the planar problem.*

This is obvious if one believes that there are "many" spatial central configurations that are not coplanar, which is still an open problem.

As pointed out in [15] a better estimate of the index of the coplanar configurations, together with (3.2) and (3.7) could solve this problem, allowing a distinction between the coplanar and spatial configurations, in the same way as it happens for the collinear configurations.

We end by showing that the collinear problem is a "repeller" in the gradient flow $\dot{q} = \nabla V|_M$.

First of all we observe that even if the manifold $M = \mathscr{E} \setminus \Delta$ is not compact, the critical points of $V|_M$ lie in a compact contained in M. Actually, it is possible to prove that there exists a neighborhood of Δ in $\mathscr{E}$ in which there are no critical points of $V|_M$ [18].

Hence the set $S = \{q \in M \,|\, \mathrm{cl}(q \cdot \mathbf{R}) \text{ is compact in } M\}$ is compact.

We set $S_0 = \{\, q \in S \mid q \text{ is a collinear configuration}\,\}$. S_0 is a compact invariant set relative to S and we can prove the following theorem.

THEOREM 3.1. S_0 *is a repeller relative to* S.[8]

PROOF. By a collinear configuration we mean a configuration where all the masses are positioned on a line through the origin in $\mathbf{R}^3$.

In [15] it has been proved that if $\bar{q}$ is a collinear central configuration the hessian $HV(\bar{q})$ is positive definite in each direction normal to $S_0 \subset S$.

The number of these normal directions is $2N - 4$. This implies that the (homotopy) index $h(\bar{q})$ relative to S is equal to

$$(3.8) \qquad\qquad h(\bar{q}, S) = h(\bar{q}, S_0) \wedge \Sigma^{2N-4},^9$$

where $h(\bar{q}, S_0)$ is the (homotopy) index of $\bar{q}$ relative to S_0.

Moreover if q is any point in S, its ω-limit, $\omega(q)$ must be a critical point of $V\mid_M$, since the flow is a gradient flow.

Thus from (3.8) we have that if $q \in S$ and $\omega(q) \in S_0$, then $q \in S_0$, because the critical points of $V\mid_M$ in S_0 are "repellers" in the direction normal to S_0.

We prove that S_0 is a repeller, relative to S, by proving that there exists a compact neighborhood N of S_0, relative to S, such that $I(N)$, the maximal invariant set in N, is S_0 and each boundary point of N leaves N in forward time.

As shown in [7] this implies that S_0 is a repeller relative to S.

Let N be a compact neighborhood of S_0 in S, disjoint from all the remaining critical points.

Thus the maximal invariant set in N is S_0. In fact if $q \in N$ and $q \cdot \mathbf{R} \in N$ then $\omega(q) \in N$ and must be a critical point of $V\mid_M$ in S_0. From (3.8) q belongs to S_0.

If $q \in \partial N$, then $q \notin S_0$ and hence $\omega(q) \notin S_0$. Then there exists $t > 0$ such that $q \cdot t \notin N$, otherwise $\omega(q)$ would belong to N and would be a critical point of $V\mid_M$ in S_0. So each boundary point of N eventually leaves N in forward time. The proof is complete. $\square$

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[8] To say that S_0 is a repeller relative to S means that there exists a neighborhood U of S_0 in S such that $\omega(U) = S_0$ [7].

[9] $\wedge$ denotes the (smash) product of two pointed spaces and Σ^{2N-4} dénotes the pointed $(2N - 4)$-sphere (for further details see [7]).

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